PMM U.S.S.R.,Vol.54,No.6,pp.727-734,1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

DECOMPOSITION AND SUBOPTIMAL CONTROL IN DYNAMICAL SYSTEMS*

F.L. CHERNOUS'KO

A non-linear controllable dynamical system described by Lagrange equations is considered. The problem of constructing bounded controlling forces which steer the system to a given state in a finite time is investigated. Sufficient conditions are indicated for the problem to be solvable. Under these conditions, the initial system splits into subsystems, each with the degree of freedom. On the basis of this decomposition, using a game-theoretic approach, a feedback control law is proposed which solves the problem posed above and is nearly time-optimal. It is shown that the control must be constructed with proper allowance for the maximum values of the non-linear terms and perturbations in the equations of motion. The perturbations may be ignored only if the ratio of the maximum level of the perturbation to that of the control does not exceed the "golden section".

1. Statement of the problem. Consider a system whose dynamics is described by the Lagrange equations

$$\frac{d}{dt}\frac{\partial T}{\partial q_k^{\cdot}}-\frac{\partial T}{\partial q_k}=Q_k'(q,q^{\cdot},t)+Q_k \tag{1.1}$$

Here $q=(q_1,\ldots,q_n)$ are generalized coordinates of the system, n is the number of degrees of freedom, dots denote differentiation with respect to time t, the generalized forces consist of controlling forces Q_k, which have to be determined, and forces Q_k' which include all other internal and external forces, including uncontrollable perturbations; throughout, $k=1,\ldots,n$. The kinetic energy of the system is

$$T=\frac{1}{2}\sum_{i,j=1}^{n}A_{ij}(q)\,q_i^{\cdot}q_j^{\cdot} \tag{1.2}$$

where A_{ij} are the elements of a symmetric positive-definite matrix $A(q)$ of order $n\times n$. Substituting (1.2) into (1.1) we reduce the equations of motion to the form

$$A(q)\,q^{\cdot\cdot}=Q+S(q,q^{\cdot},t) \tag{1.3}$$

Here $Q=(Q_1,\ldots,Q_n)$ is the vector of controlling forces, and $S=(S_1,\ldots,S_n)$ is a vector-valued function with the components

$$S_k=Q_k'-\sum_{i,j=1}^{n}\left(\frac{\partial A_{ki}}{\partial q_j}-\frac{1}{2}\frac{\partial A_{ij}}{\partial q_k}\right)q_i^{\cdot}q_j^{\cdot} \tag{1.4}$$

The controlling forces are subject to the constraints

$$|Q_k|\leqslant Q_k^{\circ} \tag{1.5}$$

where $Q_k^{\circ}>0$ are given constants.

The initial data of system (1.3) at the starting time t_0 have the form

$$q(t_0)=q^{\circ},\; q^{\cdot}(t_0)=(q^{\cdot})^{\circ} \tag{1.6}$$

and they lie in some domain D in $2n$-space: $\{q^{\circ},(q^{\cdot})^{\circ}\}\in D$.

The control problem may be formulated as follows.

Problem 1. Find a control, based on the feedback principle $Q=Q(q,q^{\cdot})$, which satisfies inequalities (1.5) and converts system (1.3) from an arbitrary initial state (1.6) in D to a

*Prikl.Matem.Mekhan.,54,6,883-893,1990

given state with zero velocities

$$q(t_1) = q^1, \quad q^\cdot(t_1) = 0 \tag{1.7}$$

in a finite time (the time $t_1 > t_0$ is not prescribed).

2. ***Simplifying assumptions.*** Let us express the matrix $A(q)$ as

$$A(q) = B(q) A_1, \quad B(q) = E + L = A(q) A_1^{-1}, \quad L = [A(q) - A_1] A_1^{-1} \tag{2.1}$$

Here A_1 is a constant symmetric positive-definite matrix of order $n \times n$, and E is the identity matrix of order $n \times n$. As $B(q)$ is a non-singular matrix, $B^{-1}(q)$ exists. Multiplying both sides of Eq.(1.3) by $B^{-1}(q)$ and using (2.1), we get

$$A_1 q^{\cdot\cdot} = Q + R(q, q^\cdot, t, Q) \tag{2.2}$$

$$R = R' + R'', \quad R' = B^{-1}(q) S(q, q^\cdot, t), \quad R'' = [B^{-1}(q) - E] Q \tag{2.3}$$

Eq.(2.2) is equivalent to the original Eq.(1.3).

It will be assumed below that the components of the vector R in (2.3) satisfy the inequalities

$$|R_k(q, q^\cdot, t, Q)| \leqslant R_k^\circ < Q_k^\circ \tag{2.4}$$

for all $t \geqslant t_0$, all $\{q, q^\cdot\} \in D$ and all Q satisfying inequalities (1.5).

The following lemma shows how to verify condition (2.4).

Lemma. Suppose that for any n-vector z and all $t \geqslant t_0$, $\{q, q^\cdot\} \in D$, the following conditions hold:

$$|A_1 z| \geqslant \mu_1 |z|, \quad |[A(q) - A_1] z| \leqslant \mu |z|$$
$$|S_k(q, q^\cdot, t)| \leqslant \nu Q_k^0, \quad 0 < \mu < \mu_1, \quad \nu > 0 \tag{2.5}$$

where μ_1, μ, ν are constants. Then the components of the vector R defined in (2.3) satisfy the following estimates for all $t \geqslant t_0$, $\{q, q^\cdot\} \in D$ and all Q satisfying (1.5):

$$|R_k(q, q^\cdot, t, Q)| \leqslant \nu Q_k^0 + \chi(1 + \nu) Q^0$$
$$\chi = \mu(\mu_1 - \mu)^{-1}, \quad Q^0 = [\Sigma (Q_k^0)^2]^{1/2} \tag{2.6}$$

Note that since the matrix A_1 is positive-definite, the constant μ_1 in (2.5) may be any positive number not exceeding its smallest eigenvalue.

Proof. By the first inequality in (2.5),

$$|A_1^{-1} z| \leqslant \mu_1^{-1} |z| \tag{2.7}$$

Here and below z is any n-vector.

It follows from (2.7) and the second inequality in (2.5), using the notation (2.1), that

$$|Lz| \leqslant \lambda |z|, \quad \lambda = \mu\mu_1^{-1} \tag{2.8}$$

It follows from the definition of B in (2.1) that

$$Bz = z + Lz \tag{2.9}$$

Inequality (2.8) and Eq.(2.9) imply the estimate

$$|Bz| \geqslant |z| - |Lz| \geqslant (1 - \lambda) |z| \tag{2.10}$$

It follows from conditions (2.5) that $\lambda < 1$. Putting $z = B^{-1} z'$ in (2.10), we obtain

$$|B^{-1} z'| \leqslant (1 - \lambda)^{-1} |z'| \tag{2.11}$$

Inequalities (2.8) and (2.11) yield

$$|LB^{-1} z| \leqslant \chi |z|, \quad \chi = \lambda(1 - \lambda)^{-1} = \mu(\mu_1 - \mu)^{-1} \tag{2.12}$$

We now rewrite (2.9), substituting there $z = B^{-1} S$. Using the resulting equality, we write the expression for R' in (2.3) in the form

$$R_k' = (B^{-1} S)_k = S_k - (LB^{-1} S)_k \tag{2.13}$$

The subscripts denote vector components. Using conditions (2.5) and inequality (2.12), we deduce from (2.13) that

$$|R_k'| \leqslant \nu Q_k^0 + |LB^{-1} S| \leqslant \nu Q_k^0 + \chi |S| \leqslant \nu Q_k^0 + \chi\nu Q^0 \tag{2.14}$$

Here we have used the notation (2.6) for Q^0. In the expression for R'' in (2.3) we substitute the expression for $B^{-1}Q$ derived from (2.9) by putting $z = B^{-1}Q$:

$$R_k'' = (B^{-1}Q - Q)_k = -(LB^{-1}Q)_k$$

Hence, using inequalities (2.12) and (1.5), we get

$$|R_k''| \leqslant |(LB^{-1}Q)_k| \leqslant |LB^{-1}Q| \leqslant \chi |Q| \leqslant \chi [\Sigma (Q_k^0)^2]^{1/2} = \chi Q^0$$

The estimates (2.6) now follow from this inequality and from (2.14), completing the proof of the lemma.

Corollary. Under the assumptions of the lemma, if $\nu < 1$ and μ is sufficiently small, then conditions (2.4) are satisfied.

In connection with the assumed constraint (2.4), the following questions arise: how should one select the matrix A_1, and how can one verify the validity of the constraint for a given system (1.3)? These questions are interrelated, because the vector R in condition (2.4) depends on A_1 through formulae (2.1) and (2.3). It follows from (2.6) and the Corollary that μ should be chosen as small as possible. Thus, A_1 should differ by as little as possible from $A(q)$ in the domain D. In other words, A_1 should be taken to be equal to some "average" value of $A(q)$ over D, e.g., to $A(q^1)$, $A(q^0)$ or $A[(q^0 + q^1)/2]$. The number μ_1 should be taken equal (or near) to the least eigenvalue of A_1 (see the remark below). Once A_1 has been chosen, if condition (2.4) fails to hold for a given system (1.3), one should, first, enlarge the domain of admissible controls, i.e., increase Q_k^0 in (1.5), in order to ensure satisfaction of the condition $\nu < 1$; and, second, one should reduce the domain D so that $A(q)$ differs only slightly from A_1.

3. *Decomposition.* Transform the variables in system (2.2) by the formula

$$A_1(q - q^1) = y \tag{3.1}$$

where q^1 is the same as in (1.7). Then system (2.2) becomes

$$y_k'' = Q_k + R_k \tag{3.2}$$

Let us assume that condition (2.4) is satisfied and that all motions of the system lie in D. Then, in view of (1.5), we have the constraints

$$|Q_k| \leqslant Q_k^0, \quad |R_k| \leqslant R_k^0 < Q_k^0 \tag{3.3}$$

After transformation (3.1), the initial conditions (1.6) and boundary conditions (1.7) become

$$y(t_0) = A_1(q^0 - q^1), \quad y^{\cdot}(t_0) = A_1(q^{\cdot})^0 \tag{3.4}$$
$$y(t_1) = 0, \quad y^{\cdot}(t_1) = 0$$

In system (3.2), which splits into n subsystems, R_k may be regarded as bounded, independent perturbations. As a result we arrive at the following theorem.

Theorem 1. Assume that condition (2.4) is satisfied and all motions of system (1.3) under consideration take place in the domain D. Then, to solve Problem 1, it suffices to solve n control problems for the linear subsystems (3.2), each with one degree of freedom. In each of these problems one must construct a scalar control $Q_k(y_k, y_k^{\cdot})$ subject to the constraint (3.3) which transfers the k-th subsystem (3.2) from an arbitrary initial state to the origin in a finite length of time, under any admissible perturbations R_k satisfying the constraint (3.3).

A different approach to the construction of controls for mechanical systems, also based on decomposition, is proposed in /1, 2/.

4. *Solution of the game problem.* Let us consider the k-th subsystem (3.2), assuming that

$$y_k = Q_k^0 x, \quad Q_k = Q_k^0 u, \quad R_k = Q_k^0 v \tag{4.1}$$

This subsystem, together with the constraints (3.3) and conditions (3.4), has the standard form

$$x'' = u + v, \quad |u| \leqslant 1, \quad |v| \leqslant \rho < 1 \tag{4.2}$$
$$x(0) = \xi, \quad x^{\cdot}(0) = \eta, \quad x(\tau) = x^{\cdot}(\tau) = 0 \tag{4.3}$$

Here

$$\rho = R_k^0/Q_k^0 < 1, \quad \xi = y_k(t_0)/Q_k^0 = [A_1(q^0 - q^1)]_k/Q_k^0, \quad \eta = y_k^{\cdot}(t_0)/Q_k^0 = [A_1(q^{\cdot})^0]_k/Q_k^0, \quad \tau = t_1 - t_0 \tag{4.4}$$

where no loss of generality is involved in putting the starting time in (4.3) equal to zero.

Employing the approach of the theory of differential games, let us find a feedback control $u(x, x^\cdot)$ which takes system (4.2) to the origin (4.3) in a minimum guaranteed time τ for any admissible perturbations v. This is a simple linear differential game for objects of the same type /3/. Its solution reduces to determining an optimal control for the system

$$x^{\cdot\cdot} = (1 - \rho) u, \quad |u| \leqslant 1, \quad \tau \to \min \tag{4.5}$$

subject to the boundary conditions (4.3). The required control $u(x, x^\cdot)$ and minimum guaranteed time τ in the game problem (4.2), (4.3) coincide with the optimal control synthesis and optimal time for problem (4.5), (4.3). Note that system (4.5) is derived from (4.2) by letting the perturbation equal $v = -\rho u$, which is the optimal control for an "opponent" who selects the perturbation v.

The solution of the time-optimal control problem (4.5), (4.2) is well-known /4/. We shall cite only those formulae necessary for our further purposes.

An optimal programmed control in problem (4.5), (4.2) assumes limiting values $u = \pm 1$ and has at most one switching point. Taking $u = \text{const}$ in the general solution of system (4.5), we obtain the equations of the phase trajectories in the $x, x^\cdot$ plane:

$$x = B' + [2(1 - \rho) u]^{-1} (x^\cdot)^2, \quad B' = \text{const} \tag{4.6}$$

The only trajectories that hit the origin as t increases are the parabolae (4.6) with $B' = 0$ and $u = \pm 1$. On these parabolae we have

$$x = (1 - \rho) u (t - \tau)^2/2, \quad x^\cdot = (1 - \rho) u (t - \tau) \quad (u = \pm 1) \tag{4.7}$$

The two branches of the parabolae (4.7) with $t \leqslant \tau$ form the switching curve (SC). The optimal control synthesis has the form

$$u(x, x^\cdot) = \operatorname{sign} \psi_\rho (x, x^\cdot) \quad \text{if } \psi_\rho \neq 0$$
$$u(x, x^\cdot) = \operatorname{sign} x = -\operatorname{sign} x^\cdot \quad \text{if } \psi_\rho = 0 \tag{4.8}$$
$$\psi_\rho (x, x^\cdot) = -x - x^\cdot |x^\cdot| [2(1 - \rho)]^{-1} \tag{4.9}$$

The equation of the SC is obtained by equating the switching function to zero: $\psi_\rho (x, x^\cdot) = 0$. All the optimal phase trajectories in the $x, x^\cdot$ plane are unions of two sections of parabolae (4.6) with $u = \pm 1$, where the second section coincides with a section of the SC (4.7) and terminates at the origin. If the initial point $\{\xi, \eta\}$ lies on the SC, there is no first section at all.

Fig.1 shows the SC $\psi_\rho = 0$ (the thick solid curve) and an optimal trajectory beginning at a point $\{\xi, \eta\}$ in the domain $\psi_\rho < 0$ (the thin solid curve). Along the first section of this trajectory $u = -1$, along the second, $u = +1$. The arrows indicate the direction of increasing time t. The field of optimal phase trajectories is centrally symmetric about the origin.

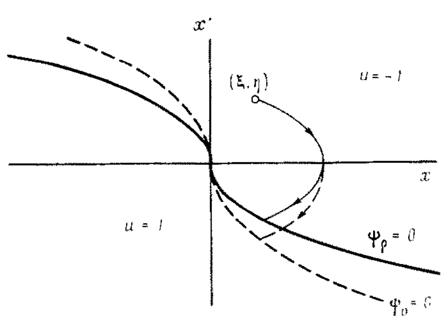

Fig.1

Let us calculate the optimal time τ necessary to reach the origin along an optimal trajectory from an arbitrary starting point $\{\xi, \eta\}$. To fix our ideas, let us assume that $\psi_\rho (\xi, \eta) < 0$. Let s denote the switching time, $s \in [0, \tau]$. The point $x(s), x^\cdot(s)$ lies, on the one hand, on the parabola corresponding to the general solution of system (4.5) with $u = -1$ which passes at $t = 0$ through the initial point $\{\xi, \eta\}$; on the other hand, it is a point of the SC (4.7) for $u = 1$. Comparing the appropriate expressions, we obtain

$$x(s) = \xi + \eta s - (1 - \rho) s^2/2 = (1 - \rho)(s - \tau)^2/2$$
$$x^\cdot(s) = \eta - (1 - \rho) s = (1 - \rho)(s - \tau)$$

Determining s and τ from these equalities, we find that

$$\tau(\xi, \eta) = (1 - \rho)^{-1} \{2 [\eta^2/2 - (1 - \rho) \xi\gamma]^{1/2} - \eta\gamma\} \tag{4.10}$$
$$\gamma = \operatorname{sign} \psi_\rho^\circ (\xi, \eta)$$

Here the symmetric property of the phase trajectories has been taken into consideration. The function ψ_ρ is that defined in (4.9). Along the SC ($\psi_\rho = 0$) the γ in (4.10) may be taken equal to either of the numbers $\gamma = \pm 1$; the value of $\tau(\xi, \eta)$ will be the same.

Formulae (4.8)-(4.10) determine an optimal control synthesis in the least guaranteed time in the game problem (4.2), (4.3). It should be observed that if the perturbation v is not optimal ($v \neq -\rho u$), the phase trajectories will also deviate from the optimal ones. However, the time required to transfer the system to the origin will not exceed the value of τ in (4.10). Note that the system, having once reached the SC, will continue to move along that curve until it reaches the origin, whatever the admissible perturbation. Under such conditions, if $v \neq -\rho u$, the result is a sliding regime of motion along the SC. Thus, if $v = 0$ on the SC, the control will take the values $u = \pm 1$ with infinitely many changes of sign, so that "on the average" $u = 1 - \rho$ or $u = -(1 - \rho)$ for the appropriate branches of the SC.

5. ***Control synthesis.*** We will now proceed to solve the original Problem 1. A synthesis of the control in this problem is derived from (4.1), (3.1):

$$Q_k(q, q^\cdot) = Q_k^0 u(x, x^\cdot) \tag{5.1}$$
$$x = y_k/Q_k^0 = [A_1(q - q^1)_k/Q_k^0$$
$$x^\cdot = y_k^\cdot/Q_k^0 = (A_1 q^\cdot)_k/Q_k^0$$

The function $u(x, x^\cdot)$ is given by formulae (4.8) and (4.9), with the parameter ρ determined for each k by (4.4). The control (5.1) is a bang-bang control, which takes the extreme admissible values $Q_k = \pm Q_k^0$.

The nature of the motion as determined by the control (5.1) may be described as follows. Let us assume first that the perturbations R_k in system (2.2) or (3.2) take the optimal "worst" values at each instant of time, i.e., those values which cause the maximum delay in bringing the system to its terminal state. In terms of system (4.2) this means that $v = -\rho u$, while in terms of system (3.2), an examination of equalities (4.1), (4.4), (5.1) yields

$$R_k = -R_k^0 u(x, x^\cdot) = -R_k^0 Q_k(q, q^\cdot)/Q_k^0 \tag{5.2}$$

Under perturbations (5.2), the motion of system (3.2) takes place, with respect to each coordinate y_k, along time-optimal trajectories of system (4.5). The transition from the original coordinates q to the coordinates y_k and to the variables $x, x^\cdot$ is given by formulae (3.1), (4.1) or (5.1).

However, if the perturbations are not the optimum ones (5.2) - as is usually the case - the phase trajectories relative to each degree of freedom, say the k-th, in the $x, x^\cdot$ plane are no longer optimal, as described in Sect.4. Under these conditions, any motion along the SC takes place in a sliding regime.

The time t_1 to steer system (1.3) (or (2.2)) to a given state (1.7) does not exceed the maximum optimal time for each of the subsystems (3.2), (4.2) or (4.5). In view of (4.4), we have

$$t_1 \leqslant t_0 + \max_{1 \leqslant k \leqslant n} \tau(\xi_k, \eta_k) \tag{5.3}$$
$$\xi_k = [A_1(q^0 - q^1)]_k/Q_k^0, \quad \eta_k = [A_1(q^\cdot)^0]_k/Q_k^0$$

The function $\tau(\xi, \eta)$ is given by (4.10) with ρ determined by (4.4).

We collect our results together in the following theorem.

Theorem 2. Assume that condition (2.4) is satisfied and that all motions under consideration take place in the domain D. Then a control synthesis $Q(q, q^\cdot)$ solving Problem 1 is given by formulae (5.1), with the function $u(x, x^\cdot)$ determined by (4.8), (4.9) and the parameter ρ for each degree of freedom by (4.4). This control brings system (1.3) to a given state (1.7) no later than the time t_1 determined by (5.3), (4.10).

The control thus constructed may be termed "suboptimal", since it is nearly a time-optimal control and is in fact made time-optimal by the "worst" perturbations.

6. ***Comparison of two control modes.*** The mode of control proposed above allows for the presence of perturbations R_k in system (2.2) or (3.2) and depends on the ratio of the perturbation and control levels (see (4.4), the parameter $\rho < 1$). Another, quite widespread, approach to control synthesis entirely ignores the perturbations, constructing the control without taking them into consideration.

The control law obtained in this way is then applied to the perturbed system. This approach is applied, for example, in the control of manipulatory robots /5, 6/.

Let us compare both modes of control as applied to system (4.2), (4.3), to which the equations of motion reduce after decomposition; our aim is to determine to what degree it is justified to ignore the perturbations when constructing the control.

When there are no perturbations $(v = 0)$, system (4.2) takes the form of (4.5) with $\rho = 0$. In that case a time-optimal control steering the system to the origin is synthesized by formulae (4.8), (4.9) with $\rho = 0$. We have

$$\psi_0(x, x^\cdot) = -x - x^\cdot \mid x^\cdot \mid / 2 \tag{6.1}$$

The SC $\psi_0 = 0$ for the case $\rho = 0$ is shown in Fig.1 (the thick dashed curve). It is the union of two branches of parabolae, differing from the branches of the SC $\psi_\rho = 0$ for

$\rho > 0$ in the coefficient only.

Let us examine the motion of system (4.2) under the control given by (4.8) and (6.1). In order to estimate the effect of the perturbations, we have to solve the following problem.

Problem 2. Find a function $v(t)$ satisfying the inequality $|v| \leqslant \rho$, such that the phase trajectory of system (4.2) under control (4.8), (6.1), with initial condition (4.3), first crosses the SC $\psi_0 = 0$ as far as possible from the origin, i.e., at the maximum possible value of $|x^\cdot|$ or, what is the same, at the maximum $|x|$.

To fix our ideas, let us suppose that the starting point $\{\xi, \eta\}$ is in the domain $\psi_0 < 0$. The phase trajectory of the system will first cross the branch of the SC $\psi_0 = 0$ on which $x > 0$, $x^\cdot < 0$. By (4.8), we then have $u = -1$ along the whole trajectory. In sum, Problem 2 reduces to the following optimal control problem:

$$x_1^\cdot = x_2, \; x_2^\cdot = -1 + v, \; |v| \leqslant \rho < 1 \tag{6.2}$$
$$x_1(0) = \xi, \; x_2(0) = \eta, \; 0 \leqslant t \leqslant \theta$$
$$2x_1(\theta) = x_2^2(\theta), \; x_2(\theta) \to \min$$

Here θ is the - as yet unknown - stopping time of the process. Applying the maximum principle /4/, we construct the Hamiltonian

$$H = p_1 x_2 + p_2(v - 1) - (v - 1), \; |v| \leqslant \rho \tag{6.3}$$

Here p_1, p_2 are the conjugate variables, which satisfy the equations $p_1^\cdot = 0$, $p_2^\cdot = -p_1$. Integrating the conjugate system, we obtain

$$p_1 = C_1, \; p_2 = C_2 - C_1 t \tag{6.4}$$

where C_1 and C_2 are arbitrary constants. In view of (6.3) and (6.4), it follows from the maximum principle that

$$v = \rho \operatorname{sign}(p_2 - 1) = \rho \operatorname{sign}(C_2 - C_1 t - 1) \tag{6.5}$$

Consequently, the optimal control $v(t) = \pm\rho$ has at most one switching point in the interval $(0, \theta)$. The transversality conditions for problem (6.2) are as follows:

$$p_1(\theta) x_2(\theta) + p_2(\theta) = 0 \tag{6.6}$$
$$H_\theta = p_1(\theta) x_2(\theta) + \rho\,|p_2(\theta) - 1| - p_2(\theta) + 1 = 0$$

Here we have used Eq.(6.5). Eliminating $p_1(\theta)$ from (6.6), we obtain

$$\rho\,|p_2(\theta) - 1| = 2p_2(\theta) - 1 \tag{6.7}$$

Let us assume first that $p_2(\theta) \geqslant 1$. Then by (6.7)

$$p_2(\theta) = (1 - \rho)(2 - \rho)^{-1} < 1$$

contrary to our assumption. Consequently, $p_2(\theta) < 1$ and, by (6.5), we have $v(\theta) = -\rho$. Thus, by (6.5), the function $v(t)$ has the form

$$v(t) = \rho, \; t \in (0, \sigma), \; 0 \leqslant \sigma < \theta \tag{6.8}$$
$$v(t) = -\rho, \; t \in (\sigma, \theta)$$

where σ is the switching time. Inserting the control (6.8) into system (6.2) and integrating with initial conditions (6.2), we obtain

$$x_1(t) = \xi + \eta t - (1 - \rho)\, t^2/2 \tag{6.9}$$
$$x_2(t) = \eta - (1 - \rho)\, t, \; t \in [0, \sigma]$$
$$x_1(t) = \xi + \eta\sigma - (1 - \rho)\,\sigma^2/2 +$$
$$[\eta - (1 - \rho)\,\sigma](t - \sigma) - (1 + \rho)(t - \sigma)^2/2$$
$$x_2(t) = \eta - (1 - \rho)\,\sigma - (1 + \rho)(t - \sigma), \; t \in [\sigma, \theta]$$

We now put $t = \theta$ in the solution (6.9) and let $x_2(\theta) = Y$. The last equality in (6.9) gives

$$\theta - \sigma = (1 + \rho)^{-1}[\eta - (1 - \rho)\,\sigma - Y] \tag{6.10}$$

We now insert (6.10) into the expression for $x_1(\theta)$ in (6.9) and use the boundary conditions (6.2) at $t = \theta$. Simplifying, we get

$$x_1(\theta) = \xi + \eta\sigma - (1-\rho)\sigma^2/2 + (1+\rho)^{-1}\{[\eta - (1-\rho)\sigma]^2 - Y^2\}/2 = Y^2/2$$

This equality yields

$$(2+\rho) Y^2 = 2(1+\rho)\xi + \eta^2 + 2\rho[2\eta\sigma - (1-\rho)\sigma^2] \tag{6.11}$$

It now remains to select the switching time σ so as to maximize $|Y|$. By (6.11), we have

$$\sigma = 0 \text{ if } \eta \leqslant 0, \ \sigma = (1-\rho)^{-1}\eta \text{ if } \eta > 0 \tag{6.12}$$

Let us calculate the optimal values of the functional $x_2(\theta) = Y$ in Problem 2. Substituting σ from (6.12) into (6.11), we obtain

$$\begin{gathered} Y = -(2+\rho)^{-1/2} 2[(1+\rho)\xi + \eta^2]^{1/2}, \ \eta \leqslant 0 \\ Y = -(1+\rho)^{1/2}(2+\rho)^{-1/2}[2\xi + (1-\rho)^{-1}\eta^2]^{1/2}, \ \eta > 0 \end{gathered} \tag{6.13}$$

The duration of the motion is found from (6.10) and (6.12):

$$\begin{gathered} \theta = (1+\rho)^{-1}(\eta - Y), \ \eta \leqslant 0 \\ \theta = (1-\rho)^{-1}\eta - (1+\rho)^{-1} Y, \ \eta > 0 \end{gathered} \tag{6.14}$$

Formulae (6.8), (6.9), (6.12)-(6.14) determine the solution of Problem 2.

Thus, if the motion begins in the domain $\psi_0 < 0, \eta \leqslant 0$, then $\sigma = 0$ and along the whole trajectory $v = -\rho$. But if the starting point $\{\xi, \eta\}$ lies in the domain $\psi_0 < 0, \eta > 0$, then by (6.12) and (6.9) we have $x_2(\sigma) = 0$, i.e., switching takes place at the same time as on the x_1 axis. Remembering that the field of the phase trajectories is symmetrical about the origin, we can express the optimal control synthesis in Problem 2 in the form

$$v(x, x^\cdot) = \rho \operatorname{sign}(x^\cdot) \tag{6.15}$$

The thin dashed curve in Fig.1 represents one of the optimal trajectories beginning at a point $\{\xi, \eta\}$ in the domain $\psi_0 < 0, \eta > 0$. It is the union of two arcs of different parabolae which touch smoothly on the x axis. The first of these arcs coincides with an arc of the optimal trajectory constructed above for the optimal synthesis with $\rho > 0$.

Assume that the starting point lies on the SC $\psi_0 = 0$, with $\eta > 0$. Then $\xi = -\eta^2/2$, and we deduce from (6.13) that

$$|Y/\eta| = \varkappa = [\rho(1+\rho)]^{1/2}[(2+\rho)(1-\rho)]^{-1/2} \tag{6.16}$$

The trajectory emanating from an arbitrary point (6.2) may be continued after crossing the SC at time θ. To that end, take $\{x(\theta), x^\cdot(\theta)\}$ as a new initial point and continue the motion as determined by system (4.2), with the control u taken from (4.8), (6.1) and the perturbation v from (6.15). The trajectory thus obtained crosses both branches of the SC $\psi_0 = 0$ an infinite number of times. The ratio between ordinate values for two consecutive switching points of the SC is equal to the number $\varkappa$ given by formula (6.16).

It is clear that the motion depends essentially on $\varkappa$. It follows from (6.16) that $\varkappa = 1$ when $\rho = \rho_*$ equals the "golden section",

$$\rho_* = (5^{1/2} - 1)/2 \approx 0.618 \tag{6.17}$$

Corresponding to values $\rho < \rho_*$ we have $\varkappa < 1$, and for $\rho > \rho_*$ we have $\varkappa > 1$.

Let us consider the nature of the possible motions of system (4.2) when the control synthesis (4.8), (6.1) corresponds to $\rho = 0$, under arbitrary admissible perturbations $|v(t)| \leqslant \rho$.

If $\rho < \rho_*, \varkappa < 1$, the trajectory of motion tends to the origin. This follows from the fact that the ordinates $|x^\cdot|$ of the points at which the trajectory cuts the SC decrease at least as quickly as a geometric progression with quotient $\varkappa < 1$ (see (6.16)). This motion reaches the origin in a finite time T_0. To estimate this time, we note that, by (4.2), in the intervals between switchings of u,

$$|\Delta x^\cdot| \geqslant (1-\rho)\Delta t \tag{6.18}$$

Here Δt is the interval between switchings of the control u, $\Delta x^\cdot$ the corresponding increment to $x^\cdot$. The increment $|\Delta x^\cdot|$ from the origin to the first switching is at most $|\eta| + |Y|$; from the first switching to the second it is, by (6.16), $|Y|(1+\varkappa)$, and so on. Continuing these estimates and using inequality (6.18), we get

$$T_0(\xi, \eta) \leqslant (1-\rho)^{-1}[|\eta| + |Y| + |Y|(1+\varkappa) \times (1 + \varkappa + \varkappa^2 + \ldots)] = (1-\rho)^{-1}[|\eta| + 2|Y|(1-\varkappa)^{-1}]$$

The quantity $|Y|$ is determined by (6.13).

If $\rho = \rho_*$, $\varkappa = 1$, the trajectory corresponding to an optimal (worst) perturbation (6.15) is periodic. It passes through the same points of the phase plane in equal time intervals T_*. To determine T_*, we put $Y = -\eta$, $\rho = \rho_*$ in (6.14). This gives

$$T_* = 2\theta = 4(1-\rho_*^2)^{-1}|\eta| = 4\rho_*^{-1}|\eta| \approx 6.464|\eta|$$

But if the perturbation v is not "worst", the trajectory will hit the origin.

If $\rho > \rho_*$, $\varkappa > 1$, the trajectory may go off to infinity given certain perturbations, e.g., for the perturbation (6.15).

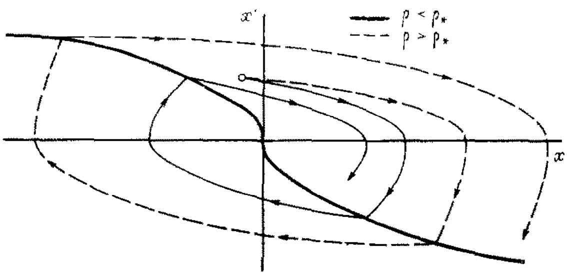

Fig.2

Fig.2 shows the phase trajectories of system (4.2) under the control (4.8), (6.1), given "worst-case" perturbations (6.15). The thick curve represents the SC $\psi_0 = 0$ (see (6.1)), the thin curves phase trajectories, with solid curves corresponding to the case $\rho < \rho_*$ and the dashed curves to $\rho > \rho_*$.

Thus, the control law (4.8), (6.1), which ignores perturbations, transfers system (4.2) to the origin in a finite time, for any admissible perturbations $|v(t)| \leqslant \rho$, only if $\rho < \rho_*$. By contrast, the control law (4.8), (4.9), constructed taking perturbations into account, transfers the same system to the origin in a finite time for any perturbations $|v(t)| \leqslant \rho$ for all $\rho < 1$.

Thus, the ratio $\rho = \rho_*$ in (6.17) is critical. When constructing a control for system (4.2), one can ignore the existence of a perturbation v only if the ratio ρ of the maximum level of the perturbation v to that of the control u remains less than the "golden section": $\rho < \rho_* \approx 0.618$.

The control method proposed in this paper is quite simple and it does not require an exact knowledge of the non-linear terms and perturbing forces in the equations of motion. All one needs is the maximum values of these quantities. The method is not overly sensitive to slight variations in the system parameters or to additional perturbations: to take such factors into consideration, one need only increase the parameter ρ, leaving a sufficient "satisfy margin" in respect of this parameter.

We observe, in conclusion, that the above method of control may be used to control the motion of manipulatory robots, since the dynamics of such systems is described by systems of equations of the form (1.3).

REFERENCES

1. PYATNITSKII E.S., Synthesis of control of manipulatory robots based on the principle of decomposition. Izv. Akad. Nauk SSSR, Tekhn. Kibern., 3, 1987.
2. PYATNITSKII E.S., The principle of decomposition in the control of mechanical systems. Dokl. Akad. Nauk SSSR, 300, 2, 1988.
3. KRASOVSKII N.N., Game Problems Involving the Convergence of Motions, Nauka, Moscow, 1970.
4. PONTRYAGIN L.S., BOLTYANSKII V.G., GAMKRELIDZE R.V. and MISHCHENKO E.F., Mathematical Theory of Optimal Processes, Nauka, Moscow, 1976.
5. AKULENKO L.D. and BOLOTNIK N.N., Synthesis of optimal control of transport motions of Manipulatory robots. Izv. Akad. Nauk SSSR, Mekh. Tverd. Tela, 4, 1986.
6. AVETISYAN V.V., AKULENKO L.D. and BOLOTNIK N.N., Optimal Control of Electrical Conductors of Industrial Robots. Akad. Nauk SSSR, Inst. Problem Mekhaniki, Moscow, 1986.

Translated by D.L.